DUSO Mathematics League 2016 - 2017

CONTEST #4.

SOLUTIONS

4 - 1. $\boxed{\mathbf{2}}$ For the sum to be prime and also greater than 2, the sum must be odd. For $29 + P$ to be odd, P must be even. Since P is an even prime, $P = \mathbf{2}$.

4 - 2. $\boxed{(\mathbf{1}, -\mathbf{2})}$ Because the solutions are b and c, the quadratic factors as $x^2 + bx + c = (x - b)(x - c)$. Expanding the right side yields $x^2 + bx + c = x^2 - (b + c)x + bc$. Comparing coefficients, $bc = c \to b = 1$, and $b = -b - c \to 1 = -1 - c \to c = -2$. The answer is $(\mathbf{1}, -\mathbf{2})$.

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4 - 3. $\boxed{\mathbf{13}}$ The third side must be greater than $11 - 7 = 4$. The third side must be less than $7 + 11 = 18$. The number of integers in the set $\{5, 6, 7, \cdots, 17\}$ is **13**.

4 - 4. $\boxed{\dfrac{\mathbf{900}}{\mathbf{11}}}$ Because a triangle has exactly one area, $3 \cdot TH = 5 \cdot MH$. Also, notice that two of the altitudes are congruent, which means two of the sides are congruent. This implies that the altitude to $\overline{TH}$ is a perpendicular bisector to $\overline{TH}$. Now, the Pythagorean Theorem yields $(TH/2)^2 + 3^2 = MH^2$, and substitution yields $\left(\dfrac{TH}{2}\right)^2 + 3^2 = \left(\dfrac{3TH}{5}\right)^2 \to \dfrac{TH^2}{4} + 9 = \dfrac{9TH^2}{25}$, which implies $(TH)^2 = \dfrac{\mathbf{900}}{\mathbf{11}}$.

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4 - 5. $\boxed{\mathbf{3}}$ Use the change of base rule to rewrite the expression as $\dfrac{\log 60}{\log 30} + \dfrac{\log 75}{\log 30} + \dfrac{\log 6}{\log 30}$. Adding and using the product property of logs, we obtain $\dfrac{\log(60 \cdot 75 \cdot 6)}{\log 30}$. Next, rewrite this as $\dfrac{\log(30^3)}{\log 30} = \dfrac{3 \log 30}{\log 30} = \mathbf{3}$.

4 - 6. $\boxed{-\mathbf{2} \text{ and } \mathbf{10} \pm \mathbf{2}\sqrt{\mathbf{35}} \textit{ [need all three]}}$ The quadratic equation will have a single root if its discriminant is zero, so solve $k^2 - 4(k + 2)(5) = 0$ to obtain $k = 10 \pm 2\sqrt{35}$. However, the equation will also have exactly one solution if the quadratic coefficient is zero and the linear coefficient is not, as happens if $k = -2$. There are three solutions: $-\mathbf{2}$ and $\mathbf{10} \pm \mathbf{2}\sqrt{\mathbf{35}}$.

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T-1. In $\triangle ABC$, the sides have lengths 5 cm, 12 cm, and 13 cm. A circle is inscribed in $\triangle ABC$. Compute the area of the circle in sq cm.

T-1Sol. $\boxed{4\pi}$ Because the lengths of the sides satisfy the Pythagorean Theorem, $\triangle ABC$ is right. The area of $\triangle ABC$ is $\frac{5 \cdot 12}{2} = 30$. The area of a triangle is equal to the product of its inradius and its semiperimeter, so $30 = r \cdot 15$ implies $r = 2$. The area of the incircle is $\pi \cdot 2^2 = \mathbf{4}\pi$.

T-2. For real numbers x and y, suppose $x + y = 5$ and $x \cdot y = 3$. Compute $x^4 + y^4$.

T-2Sol. $\boxed{\mathbf{343}}$ Factor to obtain $x^4 + y^4 = (x + y)^4 - 4x^3y - 6x^2y^2 - 4xy^3$, which is equivalent to $(x + y)^4 - 6(xy)^2 - 4xy(x^2 + y^2) = (x + y)^4 - 6(xy)^2 - 4xy((x + y)^2 - 2xy)$. The value of $x^4 + y^4$ is $5^4 - 6 \cdot 3^2 - 4 \cdot 3 \cdot (5^2 - 2 \cdot 3) = 625 - 54 - 228 = \mathbf{343}$.

T-3. Suppose that for some real x, $\cos(\sin^{-1}(\cos(\tan^{-1} x))) = \frac{1}{x}$. Compute x^2.

T-3Sol. $\boxed{\mathbf{\frac{1+\sqrt{5}}{2}}}$ First, note that $\cos(\tan^{-1} x) = \frac{1}{\sqrt{1 + x^2}}$ and that $\cos(\sin^{-1} u) = \sqrt{1 - u^2}$. Therefore, the left-hand side of the given equation simplifies to $\sqrt{1 - \frac{1}{1 + x^2}}$. Setting this equal to $\frac{1}{x}$ and solving yields $\frac{x^2}{1 + x^2} = \frac{1}{x^2} \to 1 + x^2 = x^4$, and this is a quadratic equation in x^2. Solving, $x^2 = \frac{1 + \sqrt{5}}{2}$ (rejecting the negative value of x^2).

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